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Circumferences in 1-tough graphs

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Abstract

Bauer, Morgana, Schmeichel and Veldman have conjectured that the circumference $c(G)$ of any 1-tough graph G of order $n \geqslant 3$ with minimum degree $\delta \geqslant n/3$ is at least $\min\{n, (3n+1)/4 + \delta/2\} \geqslant (11n+3)/12$. They proved that under these conditions, $c(G) \geqslant \min\{n, n/2 + \delta\} \geqslant 5n/6$. Then Bauer, Schmeichel and Veldman improved this result by getting $c(G) \geqslant \min\{n, n/2 + \delta + 1\} \geqslant 5n/6 + 1$. We show in this paper that $c(G) \geqslant \min\{n, (2n+1+2\delta)/3, (3n+2\delta-2)/4\} \geqslant \min\{(8n+3)/9, (11n-6)/12\}$.

1. Introduction and notation

We will consider only finite, undirected graphs, without loops or multiple edges. We use the notation and terminology in [4]. In particular, if G is a graph, we denote by $V(G)$ the vertex set of G, by $E(G)$ the edge set of G. For any $a \in V(G)$, $A \subseteq V(G)$, $B \subseteq V(G) - A$ and a subgraph H of G, we put

$$N_H(a) = \{v \in V(H): av \in E(G)\} \quad \text{and} \quad N(a) = N_G(a),$$

$$d_H(a) = |N_H(a)| \quad \text{and} \quad d(a) = d_G(a),$$

$$E(A,B) = \{uv \in E(G): u \in A \text{ and } v \in B\}.$$

If $C = c_1 c_2 \cdots c_p c_1$ is a cycle, we let $C[c_i, c_j]$, for $i \leqslant j$, be the subpath $c_i c_{i+1} \cdots c_j$, and $C^-[c_j, c_i] = c_j c_{j-1} \cdots c_i$, where the indices are taken modulo p. For any i and any $l \geqslant 2$, we put $c_i^+ = c_{i+1}$, $c_i^- = c_{i-1}$, $c_i^{+l} = c_{i+l}$, $c_i^{-l} = c_{i-l}$ and for any set $A \subseteq V(C)$, $A^+ = \{a^+: a \in A\}$, $A^- = \{a^-: a \in A\}$, $A^{+l} = \{a^{+l}: a \in A\}$ and $A^{-l} = \{a^{-l}: a \in A\}$. We will use similar definitions for a path. The circumference $c(G)$ is the length of a longest cycle in a graph G. A graph G is called 1-tough if the number of components

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$\omega(G - S) \leqslant |S|$ for every subset S of the vertex set $V(G)$ with $\omega(G - S) > 1$. Moreover, we denote by n the order of the graph and by δ the minimum degree of the graph.

In recent years, there have been a lot of results on the circumferences or various cycle problems in 1-tough graphs. We are interested in the following conjecture of Bauer et al., a stronger version of which was posed in [1].

Conjecture. If G is a 1-tough graph on $n \geqslant 3$ vertices with $\delta \geqslant n/3$, then $c(G) \geqslant \min\{n, (3n + 1)/4 + \delta/2\} \geqslant (11n + 3)/12$.

Examples in Fig. 1 are given in [1] to show that the bound $(3n + 1)/4 + \delta/2$, if it is correct, would be the best possible. In fact the graph in the following figure can be obtained from three disjoint subgraphs $G_1 = K_\delta$, $G_2 = K_{(3n+1)/6-\delta}$ and $G_3 = \bar{K}_{(3n-1)/6}$ (the graph of $(3n - 1)/6$ vertices without any edge) by adding all edges between G_1 and $G_2 \cup G_3$ and adding an $((3n + 1)/6 - \delta)$-matching between G_2 and G_3. This graph has a longest cycle containing δ vertices in G_1, $(3n + 1)/6 - \delta$ vertices in G_2 and $\delta + \frac{1}{2}((3n + 1)/6 - \delta)$ vertices in G_3.

In 1989, Bauer et al. [1] proved the following result.

Theorem 1 (Bauer et al. [1]). *If G is a 1-tough graph on $n \geqslant 3$ vertices with $\delta \geqslant n/3$, then $c(G) \geqslant \min\{n, n/2 + \delta\} \geqslant 5n/6$.*

Later this result was improved by Bauer et al. [2].

Theorem 2 (Bauer et al. [2]). *If G is a 1-tough graph on $n \geqslant 3$ vertices with $\delta \geqslant n/3$, then $c(G) \geqslant \min\{n, n/2 + \delta + 1\} \geqslant 5n/6 + 1$.*

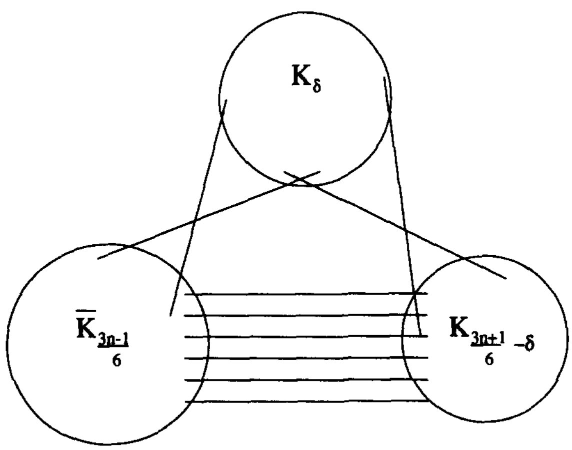

Fig. 1.

We obtain the following theorem that improves Theorems 1 and 2.

Theorem 3. *If G is a 1-tough graph on $n \geqslant 3$ vertices with $\delta \geqslant n/3$, then* $c(G) \geqslant \min\{n, (2n+1+2\delta)/3, (3n+2\delta-2)/4\} \geqslant \min\{(8n+3)/9, (11n-6)/12\}$.

We will use a theorem of Bigalke and Jung [3], which is also very important in this subject.

Theorem 4 (Bigalke and Jung [3]). *If G is a 1-tough graph on $n \geqslant 3$ vertices with $\delta \geqslant n/3$, then every longest cycle C is a dominating cycle, i.e., the vertices of $V(G) - V(C)$ form an independent set.*

2. The proof of Theorem 3

Throughout this section, let G be any 1-tough graph with $\delta \geqslant n/3$, $C = c_1 c_2 \cdots c_{n-r} c_1$ a longest cycle of G and $R = V(G) - V(C) := \{w_1, w_2, \ldots, w_r\}$.

For $\delta \geqslant (n-4)/2$, we have $n/2 + \delta + 1 \geqslant (2n+1+2\delta)/3$ and for $\delta = (n-5)/2$, we have $\lceil n/2 + \delta + 1 \rceil = \lceil (2n+1+2\delta)/3 \rceil = n-1$. Thus by Theorem 2, without loss of generality, we assume $2\delta \leqslant n-6$, and $r \geqslant 1$. By Theorem 4, R is independent. Put $X'' = \bigcup_{1 \leqslant i \neq j \leqslant r} (N(w_i) \cap N(w_j))$, $Y = \{c_j \in C\colon c_{j-1}, c_{j+1} \in X''\}$, $X' = (N(Y) \cup N(R)) - X''$ and $X = X'' \cup X'$.

We need the following lemmas.

Lemma 1. *G does not contain a path $P = p_1 p_2 \cdots p_m$ such that $N(p_1) \cup N(p_m) \subseteq V(P)$, at least one of its end-vertices, say p_1 has no consecutive neighbors on P and $m \geqslant n - r + 2$.*

Proof. Suppose that $P = p_1 p_2 \cdots p_m$ is a counterexample to the lemma. Then to avoid the existence of a cycle of at least $n - r + 1$ vertices, we deduce

$$(N_P^-(p_1) \cup N_P^{--}(p_1)) \cap N_P(p_m) = \emptyset.$$

Since p_1 has no consecutive neighbors on P, we have

$$2d(p_1) - 1 + d(p_m) + 1 \leqslant m$$

and so $m = n$. This implies that if there exists some $p_t \in N_P^{-3}(p_1) - (N_P^-(p_1) \cup N_P^{--}(p_1))$, $p_t \in N_P(p_m)$ and we have a cycle $C' = p_1 p_2 \cdots p_t p_m p_{m-1} \cdots p_{t+3} p_1$ of $n-2$ vertices such that $p_{t+1} p_{t+2} \in E(G - V(C))$, which by Theorem 4 implies $c(G) \geqslant |V(C')| + 1 \geqslant n - 1$, a contradiction. So $N_P^{-3}(p_1) - (N_P^-(p_1) \cup N_P^{--}(p_1)) = \emptyset$, and if $j = \max\{i\colon p_i p_1 \in E(G)\}$, we have $N_P(p_m) = \{p_j, p_{j+1}, \ldots, p_{m-1}\}$ and $N_P(p_1) = \{p_2, p_4, \ldots, p_j\}$. By 1-toughness, there exists some path between

$\{p_1, p_2, \ldots, p_{j-1}\}$ and $\{p_{j+1}, p_{j+2}, \ldots, p_m\}$ and such path creates a cycle of at least $n-1$ vertices, a contradiction. □

Corollary 1. *G does not contain a path* $P = p_1 p_2 \cdots p_{n-r}$ *such that*

(a) $V(P) = V(C)$

(b) $p_1, p_{n-r} \in X$

(c) *if* $p_1 \notin N(R)$ *then for a vertex* $p_m \in Y \cap N(p_1)$, *the vertices* $p_{m-1}, p_{m+1} \in X''$. *And if* $p_{n-r} \notin N(R)$ *then for a vertex* $p_t \in Y \cap N(p_{n-r})$, *the vertices* $p_{t-1}, p_{t+1} \in X''$.

(d) *P has no consecutive vertices which are both in* $N(R)$.

Proof. Suppose that $P = p_1 p_2 \cdots p_{n-r}$ is a counterexample to the corollary.

Then, without loss of generality, we have the following three cases.

Case 1: $p_1 w_1, p_{n-r} w_2 \in E(G)$ for some vertices $w_1, w_2 \in R$. Then $w_1 p_1 p_2 \cdots p_{n-r} w_2$ is a cycle longer than C when $w_1 = w_2$ or a path of $n - r + 2$ vertices when $w_1 \neq w_2$, which contradicts Lemma 1.

Note that we do not need the condition (c) in this case.

It is clear from this case that there do not exist two consecutive vertices of $N(R)$ on the cycle C. Hence $Y \cap N(R) = \emptyset$.

Case 2: $p_1 \in N(R)$ and $p_{n-r} \notin N(R)$. By the definition of X, $p_{n-r} \in N(p_t)$ for some $p_t \in Y$. Clearly $p_t \neq p_1$ since $Y \cap N(R) = \emptyset$. By (c), p_{t-1}, $p_{t+1} \in X''$. Then the path $P' = p_1 p_2 \cdots p_t p_{n-r} p_{n-r-1} \cdots p_{t+1}$ is of the type excluded in case 1 since $p_{n-r} \notin N(R)$ and so there are no consecutive vertices of $N(R)$ on P'.

From this case we deduce that $Y \cap X = \emptyset$.

Case 3: $p_1, p_{n-r} \notin N(R)$. By the definition of X, $p_1 p_m$, $p_{n-r} p_t \in E(G)$ for some $p_m, p_t \in Y$. Then $p_t \neq p_1$ and $p_m \neq p_{n-r}$ since $Y \cap X = \emptyset$. It follows by (c) that $p_{t-1}, p_{t+1}, p_{m-1}, p_{m+1} \in X''$. If $m \leqslant t$ let

$$Q' = p_{m-1} p_{m-2} \cdots p_1 p_m p_{m+1} \cdots p_t p_{n-r} p_{n-r-1} \cdots p_{t+1}$$

and if $t \leqslant m-1$ let

$$Q'' = p_{t-1} p_{t-2} \cdots p_1 p_m p_{m-1} \cdots p_t p_{n-r} p_{n-r-1} \cdots p_{m+1}.$$

Then both Q' and Q'' are of the type excluded in Case 1 since $p_1, p_{n-r} \notin N(R)$ and so there are no consecutive vertices of $N(R)$ on Q' or Q''.

These contradictions complete the proof. □

Corollary 2. (a) $X \subset V(C)$.

(b) *X does not contain consecutive vertices on C.*

(c) X^+ *and* X^- *are independent sets.*

Proof. (a) and (b) are clear from Corollary 1 and its proof. If there exist two adjacent vertices $c_i, c_j \in X^+$, then $c_i, c_j \notin Y$, Otherwise one of them should be in $X \cap X^+$ contrary to (b). Since $c_i, c_j \notin X$, then $c_i, c_j \notin N(R)$. But the path

$C^-[c_i^-, c_j^+]c_jc_iC[c_i^+, c_j^-]$ contradicts Corollary 1. So X^+ is independent. Similarly X^- is independent. □

Lemma 2. *G does not contain two disjoint paths $P' = p_1p_2 \cdots p_m$ and $P'' = p_{m+1}p_{m+2} \cdots p_s$ such that $N(p_1) \cup N(p_m) \cup N(p_{m+1}) \cup N(p_s) \subseteq V(P' \cup P'')$, p_1p_m, $p_{m+1}p_s \notin E(G)$, at least three of their end-vertices, say p_1, p_m and p_{m+1} have no consecutive neighbors, respectively, on P' nor on P'' and $s \geqslant n - r + 3$.*

Proof. Suppose that there exist such P' and P''. Since p_1, p_m, p_{m+1} have no consecutive neighbors on P, respectively, by Lemma 1, $p_1p_{m+1}, p_1p_{m+2} \notin E(G)$ and $p_sp_m, p_sp_{m-1} \notin E(G)$. Then to avoid the existence of a path that satisfies the conditions of Lemma 1, we deduce that for $P \in \{P', P''\}$

$$(N_P^-(p_1) \cup N_P^{--}(p_1)) \cap N_P(p_s) = \emptyset.$$

It follows that

$$2|N_{P'}(p_1)| - 1 + |N_{P'}(p_s)| + 2 + 2|N_{P''}(p_1)| + |N_{P''}(p_s)| + 1 \leqslant s$$

and so

$$s \geqslant 2d(p_1) + d(p_s) + 2 \geqslant n + 2,$$

a contradiction. □

Under our definitions of X', X'', X and Y, we have the following.

Lemma 3. *$\{c_ic_j \in E(G)\colon c_i \in X^+, c_j \in X''^-, \text{and } C[c_i, c_j] \cap X \neq \emptyset\} = \emptyset$.*

Proof. Suppose that there are $c_i \in X^+$ and $c_j \in X''^-$ such that $c_ic_j \in E(G)$ and there exists some $c_t \in X$ with $i \leqslant t \leqslant j$. Then we have two paths $P' = C[c_j^+, c_i^-] := p_1p_2 \ldots p_m$ and $P'' = C^-[c_t, c_i^+]c_ic_jC^-[c_j^-, c_{t+1}] := p_{m+1}p_{m+2} \cdots p_{n-r}$. Then $V(P' \cup P'') = V(C)$. We have the following cases.

Case 1: p_1, p_m, p_{m+1} $(c_j^+, c_i^-, c_t) \in N(R)$. Let $w_1p_1, w_2p_m, w_3p_{m+1} \in E(G)$ for some $w_1, w_2, w_3 \in R$ such that $w_1 \neq w_2$ which is possible by the definition of X''. If w_3 is equal to one of w_2 and w_1, say w_2, then the path $Q = w_1P'[p_1, p_m]w_2P''[p_{m+1}, p_{n-r}]$ contradicts Lemma 1. If w_1, w_2, w_3 are all different, P' and P'' contradict Lemma 2.

Case 2: $p_1, p_m \in N(R)$ and $p_{m+1}p_w \in E(G)$ for some $p_w \in Y$. By (c), $p_{w-1}, p_{w+1} \in X''$. Then we define

$$Q' = \begin{cases} P' & \text{if } w \geqslant m + 1, \\ p_1p_2 \cdots p_{w-1} & \text{if } w \leqslant m - 1 \end{cases}$$

and

$$Q'' = \begin{cases} p_{w-1}p_{w-2} \cdots p_{m+1}p_wp_{w+1} \cdots p_{n-r} & \text{if } w \geqslant m + 1, \\ p_mp_{m-1} \cdots p_wp_{m+1}p_{m+2} \cdots p_{n-r} & \text{if } w \leqslant m - 1. \end{cases}$$

Then three of the four end-vertices of Q' and Q'' are in $N(R)$ and we obtain contradictions as in case 1.

Case 3: $p_1, p_{m+1} \in N(R)$ and $p_m p_q \in E(G)$ for some $p_q \in Y$. By (c), $p_{q-1}, p_{q+1} \in X''$. Then we define

$$Q' = \begin{cases} p_1 p_2 \cdots p_q p_m p_{m-1} \cdots p_{q+1} & \text{if } q \leqslant m-1, \\ p_1 p_2 \cdots p_m p_q p_{q-1} \cdots p_{m+1} & \text{if } q \geqslant m+1 \end{cases}$$

and

$$Q'' = \begin{cases} P'' & \text{if } q \leqslant m-1, \\ p_{q+1} p_{q+2} \cdots p_{n-r} & \text{if } q \geqslant m+1. \end{cases}$$

Then three of the four end-vertices of Q' and Q'' are in $N(R)$ and we obtain contradictions as in case 1.

Case 4: $p_1 \in N(R)$ and $p_m p_w, p_{m+1} p_q \in E(G)$ for some $p_q, p_w \in Y$. By (c), $p_{q-1}, p_{q+1}, p_{w-1}, p_{w+1} \in X''$. Then we define

$$Q' = \begin{cases} p_1 p_2 \cdots p_{q-1} & \text{if } q \leqslant w \leqslant m-1, \\ p_1 p_2 \cdots p_w p_m p_{m-1} \cdots p_{q+1} & \text{if } w \leqslant q-1 \leqslant m-2, \\ p_1 p_2 \cdots p_w p_m p_{m-1} \cdots p_{w+1} & \text{if } w \leqslant m-1 \leqslant q-3, \\ p_1 p_2 \cdots p_q p_{m+1} p_{m+2} \cdots p_{w-1} & \text{if } q+3 \leqslant m+2 \leqslant w, \\ p_1 p_2 \cdots p_m p_w p_{w+1} \cdots p_q p_{m+1} p_{m+2} \cdots p_{w-1} & \text{if } m+1 \leqslant w \leqslant q, \\ p_1 p_2 \cdots p_m p_w p_{w-1} \cdots p_q p_{m+1} p_{m+2} \cdots p_{q-1} & \text{if } m+1 \leqslant q \leqslant w-1 \end{cases}$$

and

$$Q'' = \begin{cases} p_{w+1} p_{w+2} \cdots p_m p_w p_{w-1} \cdots p_q p_{m+1} p_{m+2} \cdots p_{n-r} & \text{if } q \leqslant w \leqslant m-1, \\ p_{w+1} p_{w+2} \cdots p_q p_{m+1} p_{m+2} \cdots p_{n-r} & \text{if } w \leqslant q-1 \leqslant m-2, \\ p_{q-1} p_{q-2} \cdots p_{m+1} p_q p_{q+1} \cdots p_{n-r} & \text{if } w \leqslant m-1 \leqslant q-3, \\ p_{q+1} p_{q+2} \cdots p_m p_w p_{w+1} \cdots p_{n-r} & \text{if } q+3 \leqslant m+2 \leqslant w, \\ p_{q+1} p_{q+2} \cdots p_{n-r} & \text{if } m+1 \leqslant w \leqslant q, \\ p_{w+1} p_{w+2} \cdots p_{n-r} & \text{if } m+1 \leqslant q \leqslant w-1. \end{cases}$$

Then three of the four end-vertices of Q' and Q'' are in $N(R)$ and we obtain contradictions as in case 1.

The lemma is proved. $\square$

We now prove the theorem.

Clearly we have $|X| = |X^+| = |X^-|$ and $|X''| = |X''^+| = |X''^-|$. Let $x = |X|$, $x' = |X'|$ and $x'' = |X''|$. Since $R \cup X^+$ is independent, by 1-toughness we have $r + x \leqslant n/2$. If $r \leqslant (n - 2\delta + 2)/4$, it follows that $c(G) = n - r \geqslant (3n + 2\delta - 2)/4$, as required.

Suppose $r > (n - 2\delta + 2)/4$. We claim that

$$x'' \geqslant \delta. \tag{1}$$

To prove the claim, assume that $x'' \leqslant \delta - 1$. Then any vertex in R has at least one neighbor in X'. It gives that $x \geqslant \delta - 1 + r$ and hence $\delta - 1 + r \leqslant n/2 - r$ which gives a contradiction of the assumption of r. So (1) holds.

Consider the subgraph $G[R \cup X^+ \cup X''^-]$. By Corollary 2(c) and Lemma 3, $E(G[R \cup X^+ \cup X''^-]) \subseteq \{c_i c_j \in E(G)\colon\ c_i \in X^+,\ c_j \in X''^- \textit{and}\ C[c_i, c_j] \cap X = \emptyset\}$. So $G[R \cup X^+ \cup X''^-]$ contains at least $r + x$ connected components. Since G is 1-tough, we have

$$r + x' + x'' \leqslant n - r - x' - x'' - |X''^- - X^+|$$

and so

$$2r \leqslant n - 2x' - 2x'' - (x'' - y - x') = n - 3x'' - x' + y. \tag{2}$$

Since $N(R) \cup N(Y) \subseteq X$ and $R \cup Y$ is independent, by 1-toughness, we have

$$r + y + 1 \leqslant x = x' + x''. \tag{3}$$

Adding (2) and (3) and then applying (1) give

$$3r \leqslant n - 2x'' - 1 \leqslant n - 2\delta - 1.$$

We have $r \leqslant (n - 2\delta - 1)/3$ and $c(G) \geqslant (2n + 2\delta + 1)/3$, as required.

The proof of the theorem is completed.

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